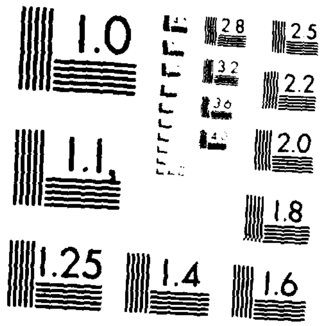
AD-A131 795 KINGMAN'S SUBADDITIVE ERGODIC THEOREM(U) STANFORD UNIV CA DEPT OF STATISTICS J M STEELE 03 SEP 82 TR-324 N00014-76-C-0475 1/1

UNCLASSIFIED F/G 12/1 NL

END
DATE FILMED
9-83
DTIC




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS [illegible]

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>324 | 2. GOVT ACCESSION NO.<br>AD-A131 785 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>KINGMAN'S SUBADDITIVE ERGODIC THEOREM | | 5. TYPE OF REPORT & PERIOD COVERED<br>TECHNICAL REPORT |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>J. Michael Steele | | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-76-C-0475 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Statistics<br>Stanford University<br>Stanford, CA 94305 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>NR-042-267 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Office Of Naval Research<br>Statistics & Probability Program Code 411SP<br>Arlington, VA 22217 | | 12. REPORT DATE<br>September 3, 1982 |
| | | 13. NUMBER OF PAGES<br>6 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ergodic theorem; subadditive ergodic theorem.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A simple proof of Kingman's subadditive ergodic theorem is given.

-- 1 - AGENCY ACCESSION NO: DN623421
-- 2 - DATE OF SUMMARY: 19 JAN 82
-- 3 - DATE OF PREV. SUMMARY: 31 AUG 81
-- 4 - KIND OF SUMMARY: CHANGE
-- 5 - SUMMARY SECURITY IS: UNCLASSIFIED
-- 6 - SECURITY OF WORK: UNCLASSIFIED
-- 8A1 - DIST. LIMITATION: UNLIMITED
-- 8B - CONTRACTOR ACCESS: YES
--10A1 - PRIMARY PROGRAM ELEMENT: 61153N
--10A2 - PRIMARY PROJECT NUMBER: RR01405
--10A2A - PRIMARY PROJECT AGENCY AND PROGRAM: RR01405
--10A3 - PRIMARY TASK AREA: RR0140501
--10A4 - WORK UNIT NUMBER: NR-042-267
--10B1 - CONTRIBUTING PROGRAM ELEMENT (1ST): NSA
--10B2 - CONTRIBUTING PROJECT NUMBER (1ST): RR01405
--10B3 - CONTRIBUTING TASK AREA (1ST): RR0140501
--10C1 - CONTRIBUTING PROGRAM ELEMENT (2ND): NAVSEA
-- 11 - TITLE: (U) ANALYTIC TECHNIQUES FOR MILITARY STATISTICS PROBLEMS
-- 11A - TITLE SECURITY: U
-- 12 - S + T AREAS:
-- 009700 MATHEMATICS AND STATISTICS
-- 011700 OPERATIONS RESEARCH

-

-- 009100 LOGISTICS
-- 13 - WORK UNIT START DATE: JUN 74
-- 14 - ESTIMATED COMPLETION DATE: CO
-- 15A - PRIMARY FUNDING AGENCY: NAVY
-- 15B - OTHER FUNDING AGENCY: NATIONAL SECURITY AGENCY
-- 15C - OTHER FUNDING AGENCY: NAVY
-- 16 - PERFORMANCE METHOD: CONTRACT
--17A1 - CONTRACT/GRANT EFFECTIVE DATE: MAY 76
--17A2 - CONTRACT/GRANT EXPIRATION DATE: MAR 83
-- 17B - CONTRACT/GRANT NUMBER: N00014-76-C-0475
-- 17C - CONTRACT TYPE: COST TYPE

-- 19A - DOD ORGANIZATION:      OFFICE OF NAVAL RESEARCH (CODE 411(SP)
-- 19B - DOD ORG. ADDRESS:      ARLINGTON, VA. 22217
-- 19C - RESPONSIBLE INDIVIDUAL:      WEGMAN, E J

-
-- 19D - RESPONSIBLE INDIVIDUAL PHONE: 202-696-4315
-- 19U - DOD ORGANIZATION LOCATION CODE: 5110
-- 19S - DOD ORGANIZATION SORT CODE: 35832
-- 19T - DOD ORGANIZATION CODE: 265250
-- 20A - PERFORMING ORGANIZATION:      STANFORD UNIV DEPT OF STATISTICS
-- 20B - PERFORMING ORG. ADDRESS:      STANFORD CALIF 94305
-- 20C - PRINCIPAL INVESTIGATOR:      SOLOMON, H
-- 20D - PRINCIPAL INVESTIGATOR PHONE: 415-497-1185
-- 20U - PERFORMING ORGANIZATION LOCATION CODE: 0612
-- 20N - PERF. ORGANIZATION TYPE CODE: 1
-- 20S - PERFORMING ORG. SORT CODE: 44778
-- 20T - PERFORMING ORGANIZATION CODE:      332580
-- 21E - MILITARY/CIVILIAN APPLICATIONS:      CIVILIAN
-- 22 - KEYWORDS: (U) STOCHASTIC PROCESSES ;(U) WEAPONRY ;(U) SYSTEM
-- RELIABILITY;(U) ANALYSIS OF VARIANCE ;(U) APPLIED PROBABILITY ;
-- 23 - TECHNICAL OBJECTIVE: (U) TO IMPROVE THE NAVY'S WEAPONRY BY CREATING
-- IMPROVED STATISTICAL TECHNIQUES FOR THE OBJECTIVE EVALUATION AND TESTING
-- OF WEAPONS SYSTEMS.
-- 24 - APPROACH: (U) TO CONDUCT RESEARCH ON A VARIETY OF STATISTICAL
-- PROBLEMS INCLUDING GOODNESS-OF-FIT STATISTICS, CURVE FITTING,
-- DIRECTIONAL DATA, VARIANCE REDUCTION, POISSON PROCESSES, MULTIVARIATE
-- DATA ANALYSIS, PATTERN RECOGNITION AND RANDOM TRANSFORMATIONS ON A
-- FINITE SET.

-
-- 25 - PROGRESS: (U) PROGRESS HAS BEEN MADE ON SEVERAL STATISTICS PROBLEMS
-- INCLUDING (A) DEVELOPMENT OF A SYSTEMATIC APPROACH FOR PROVIDING
-- GOODNESS-OF-FIT TESTS AS WELL AS A GENERAL TECHNIQUE FOR PRODUCTING
-- INVARIANT CHARACTERIZATIONS, (B) DERIVATIONS OF EXACT AND ASYMPTOTIC
-- RESULTS DESCRIBING THE CYCLIC BEHAVIOR OF RANDOM TRANSFORMATIONS ON A
-- FINITE SET AND (C) INVESTIGATIONS OF RANDOM PACKING STUDIES IN TWO,
-- THREE AND HIGHER DIMENSIONS AND ANALYSES OF DISTRIBUTIONS OF AREAS,
-- PERIMETERS AND SIDES OF RANDOM POLYGONS IN THE PLANE. (BLAISDELL, B.
-- EDWIN AND HERBERT SOLOMON. 'RANDOM SEQUENTIAL PACKING IN EUCLIDEAN
-- SPACES OF DIMENSIONS THREE AND FOUR AND A CONJECTURE OF PALISTI,' TR NO.
-- 304, JULY 1981.)
-- 30 - SUBELEMENT CODE: 14
-- 37 - DESCRIPTORS: (U) *MILITARY RESEARCH ;(U) *MATHEMATICAL MODELS ;
-- (U) MATHEMATICS LABORATORIES;(U) OPERATIONS RESEARCH ;(U)
-- RESEARCH MANAGEMENT ;(U) *STATISTICAL ANALYSIS ;
-- 39 - PROCESSING DATE (RANGE): 13 JAN 82
-- 43 - CUMULATIVE PARTIAL CONTRACT/GRANT CODE: 0
--******

-- <<ENTER NEXT COMMAND>>

# KINGMAN'S SUBADDITIVE ERGODIC THEOREM

By

J. Michael Steele

TECHNICAL REPORT NO. 324

September 3, 1982

Prepared Under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director



DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Kingman's Subadditive Ergodic Theorem

By

J. Michael Steele

The objective of this note is to give a proof of Kingman's subadditive ergodic theorem which is perhaps simpler and more direct than those previously given.([2], [3], [4], [5], [6], [7], [11]).

Theorem. Suppose $T$ is a measure preserving transformation of the probability space $(\Omega,\mathfrak{F},\mu)$ and that $\{g_n, 1 \le n < \infty\}$ is a sequence of integrable functions which satisfy

$$(1) \qquad g_{n+m}(x) \le g_n(x) + g_m(T^n x) .$$

With probability one we then have the existence of the limit

$$\lim_{n\to\infty} g_n(x)/n = g(x) \ge -\infty ,$$

where $g(x)$ is an invariant function.

Proof. We first check that $g(x) = \liminf g_n(x)/n$ is an invariant function. Since $g_{n+1}(x)/n \le g_1(x)/n + g_n(Tx)/n$ we see $g(x) \le g(Tx)$ which gives $\{g(x) > \alpha\} \subset T^{-1}\{g(x) > \alpha\}$. The fact that $T$ is measure preserving then implies $\{g(x) > \alpha\} = T^{-1}\{g(x) > \alpha\}$ up to null sets. This implies $g$ is measurable with respect to the invariant $\sigma$-field and hence is invariant. The function $\phi(x) = \max(t,g(x))$ where $t \in (-\infty,0)$ is also invariant.

For $\varepsilon > 0$, set $A_\ell = \{x: g_\ell(x) \leq \ell(\phi(x)+\varepsilon)\}$ and note that $\mu(\bigcup_{\ell=1}^{\infty} A_\ell) = 1$, so we can choose $N$ such that for $B(N) = (\bigcup_{\ell=1}^{N} A_\ell)^c$ we have $\mu(B(N)) \leq \varepsilon$.

Now, by Birkhoff's ergodic theorem, $\frac{1}{n}\,\Sigma_{k=1}^{n}\, 1_{B(N)}(T^k x)$ converges a.s. to $E(1_{B(N)}|\mathcal{G})$ where $\mathcal{G}$ is the invariant field of $T$; so by Chebyshev's inequality

$$\mu(\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{B(N)}(T^k x) \geq \lambda) \leq \varepsilon/\lambda \ .$$

Setting

$$C_M = \{x: \sum_{k=1}^{n} 1_{B(N)}(T^k x) \leq 2n\lambda, \quad \forall n \geq M\}$$

we have for $M$ sufficiently large that $\mu(C_M) \geq 1-2\varepsilon/\lambda$.

For any $x \in C_M$ and $n \geq M$ we obtain a decomposition for the integer set $[0,n)$ into three classes of intervals by the following algorithm:

Begin with $k = 0$. If $k$ is the least integer in $[0,n)$ not in an interval already constructed then we consider $T^k x$. If $T^k x \in B(N)^c$ then there is an $\ell \leq N$ so that $g_\ell(T^k x) \leq \ell(\phi(T^k x) + \varepsilon) \leq \ell(\phi(x) + \varepsilon)$ and we take $[k, k+\ell)$ as an element of our decomposition provided $k+\ell \leq n$. If $T^k x \in B(N)$ we take the singleton interval $[k,k+1)$.

This algorithm provides a decomposition of some $[0,n')$ with $n'-N \leq n' \leq n$, and it is extended to a decomposition of $[0,n)$ by adding as many singletons as necessary.

Thus for any $x \varepsilon C_M$ we have a decomposition of $[0,n)$ into a set of $u$ intervals $[\tau_i, \tau_i+\ell_i)$, $1 \le \ell_i \le N$, for which $g_{\ell_i}(T^{\tau_i}x) \le \ell_i(\phi(x)+\varepsilon)$ together with a set of $v$ singletons $[\sigma_i, \sigma_i+1)$ for which $1_{B(N)}(T^{\sigma_i}x) = 1$, and a set of $w$ singletons contained in $(n-N,n)$.

By (1) and this decomposition of $[0,n)$ we have on $C_M$,

$$(2) \qquad g_n(x) \le \sum_{i=1}^{u} g_{\ell_i}(T^{\tau_i}x) + \sum_{i=1}^{v} g_1(T^{\sigma_i}x) + \sum_{i=1}^{w} g_1(T^{n-i}x)$$

$$\le (\phi(x)+\varepsilon) \sum_{i=1}^{u} \ell_i + \sum_{i=1}^{n} g_1(T^k x)1_{B(N)}(T^k x) + \sum_{i=1}^{N} |g_1(T^{n-i}x)| \ .$$

Since

$$\sum_{k=1}^{\infty} \mu(|g_1(T^k x)| > \delta k) = \sum_{k=1}^{\infty} \mu(|g_1(x)| > \delta k) < \infty \text{ , for all } \delta > 0,$$

the Borel-Cantelli lemma implies $g_1(T^k x)/k \to 0$ a.s.. From this one easily sees that almost surely

$$(3) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{N} |g_1(T^{n-i}x)| = 0 \ .$$

Also, by Birkhoff's ergodic theorem we have

$$(4) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g_1(T^k x)1_{B(N)} = E(g_1 1_{B(N)}|\mathcal{G}) \ .$$

Finally $n \geq \Sigma_{i=1}^{u} \ell_i \geq n - N - 2\varepsilon n$ so from (2), (3), (4) we have on $C_M$ that

(5) $$\limsup_{n \to \infty} g_n(x)/n \leq \phi(x) + 3\varepsilon + E(g_1 1_{B(N)} | \mathcal{G}) .$$

For $N \to \infty$, $1_{B(N)} \to 0$ a.s. so by dominated convergence $E(g_1 1_{B(N)} | \mathcal{G}) \to 0$ a.s.. Therefore, by the arbitrariness of $\varepsilon$, $t$, $\lambda$, $N$, and $M$ we have with probability one that

$$\limsup_{n \to \infty} g_n(x)/n \leq \liminf_{n \to \infty} g_n(x)/n ,$$

which completes the proof of convergence.

Remarks. (1). The preceeding proof was motivated by the recent proofs of the Birkhoff ergodic theorem and the Shannon-MacMillan-Breiman theorem given by Paul Shields [8]. That work is in part devoted to the simplification and exposition of some recent work of Ornstein and Weiss [7].

(2). Inspection of the preceeding proof shows that it suffices to assume that just $g_1^+ \in L^1$, instead of $g_n \in L^1$, for all $n$. That the subadditive ergodic theorem persists under this condition was already observed in Kingman [5, p. 885].

(3). David Aldous has shown that Kingman's subadditive ergodic theorem can be used to give a very brief proof of the ergodic theorem for Banach space due to Maurier [8]. If $\{X_i\}$ is a stationary process with values in a Banach space $F$, we first note there is no loss in assuming $E(X_1 | \mathcal{G}) = 0$ where $\mathcal{G}$ is the invariant $\sigma$-algebra. Also, we can find a linear operator $\theta$ on $F$ with finite dimensional range such that $E\|X_1 - \theta X_1\| \leq \varepsilon$. Now

Birkhoff's ergodic theorem (applied to linear functionals) shows that $\frac{1}{n} \Sigma_{i=1}^{n} \theta(X_i)$ converges a.s. and in $L^1$ to $E\theta(X_1)$. The $L^1$ convergence guarantees $\overline{\lim}\ E\|\frac{S_n}{n} - E\theta(X_1)\| \leq \varepsilon$ from which it follows that $\lim E\|S_n/n\| = 0$. But since $\|S_n\|$ is a subadditive process $\|S_n/n\|$ converges a.s., and now necessarily converges a.s. to zero.

(4) Andrés del Junco has pointed out that there is a useful device of Akcoglu and Sucheston [1] which can be used to circumvent the estimations of the last two terms in equation (2). The idea is that $g_m' = g_m(x) - \Sigma_{i=0}^{m-1} g_1(T^i x)$ defines a (negative) subadditive process. The last two terms in equation (2) applied to $g_m'$ would then simply not appear.

The proof given above was retained in order to maximize conceptual simplicity (at the cost of a little extra computation).

Acknowledgement. I would like to thank Paul Shields for making available to me his manuscript [10].). I also owe a debt David Aldous, Andrés del Junco and Joe Marhoul for their comments on an earlier draft and their permission to incorporate the remarks given above.

## References

[1] Akcoglu, M.A. and Sucheston, L. (1978). A ratio ergodic theorem for superadditive processes. Z. Wahrscheinlichkeitstheorie and Verw. Gebiete, 44, 269-278.

[2] Burkholder, D.L. (1968). Contributions to discussion of [5].

[3] Del Junco A. (1977). On the decomposition of a subadditive stochastic process. Ann. Prob., 5, 298-302.

[4] Derriennic, Y. (1975). Sur le theorem ergodique sous-add f. C.R. Acad. Sci. Paris Ser. A., 281, 985-988.

[5] Kingman, J.F.C. (1968). The ergodic theory of subadditi stochastic processes. J. Roy. Statist. Soc. Ser. B, 30, 499-510.

[6] ______ (1973). Subadditive ergodic theory. Ann. Prob., 1, 883-905.

[7] ______ (1976). Subadditive processes. Ecole d'Eté de Probabilités de Saint-Flour. Lecture Notes in Math. 539, 168-223, Springer, Berlin.

[8] Mourier, E. (1953). Elements aleatoire dans un espace de Banach. Ann. Inst. H. Poincaré Sect. B, 13, 161-244.

[9] Ornstein, D. and Weiss, B. (1980). The Shannon-McMillan-Breiman Theorem for Amenable Groups, to appear.

[10] Shields, P. (1980). The ergodic and entropy theorems revisited (to appear in IEEE, Transactions in Information Theory).

[11] Smeltzer, M.D. (1977). Subadditive stochastic processes. Bull. Amer. Math. Soc., 83, 1054-1055.

ATE
LMED
8